

Edge distribution of graphs with few induced copies of a given graph

V. Nikiforov

Department of Mathematical Sciences
University of Memphis, Memphis, TN 38152
e-mail: vnikifrv@memphis.edu

November 26, 2003

Abstract

We show that if a simple graph contains few induced copies of a given graph, then its edges are distributed rather unevenly.

More precisely, for all $\varepsilon > 0$ and $r \geq 2$, there exist $\xi = \xi(\varepsilon, r) > 0$ and $L = L(\varepsilon, r)$ such that, for every graph H of order r , and every graph G of sufficiently large order n , the following assertion holds.

If G contains fewer than ξn^r copies of H , then there exists a partition $V(G) = \cup_{i=0}^q V_i$ with $|V_0| < q \leq L$, such that $|V_i| = \lfloor n/q \rfloor$, and

$$e(V_i) < \varepsilon \binom{|V_i|}{2} \quad \text{or} \quad e(V_i) > (1 - \varepsilon) \binom{|V_i|}{2}$$

for every $i \in [q]$.

In particular, for all $\varepsilon > 0$ and $r \geq 2$, there exist $\xi = \xi(\varepsilon, r) > 0$ and $L = L(\varepsilon, r)$ such that, for every graph G of sufficiently large order n , the following assertion holds.

If G has fewer than ξn^r r -cliques, then there exists a partition $V(G) = \cup_{i=0}^q V_i$ with $|V_0| < q \leq L$ such that

$$|V_i| = \lfloor n/q \rfloor, \quad \text{and} \quad e(W_i) < \varepsilon \binom{|V_i|}{2}$$

for every $i \in [q]$.

We derive also a number of related results.

1 Introduction

Our graph-theoretic notation is standard (e.g., see [5]); thus we write $G(n, m)$ for a graph of order n and size m . Given two graphs H and G we write $k_H(G)$ for the number of induced copies of H in G ; $k_r(G)$ stands for $k_{K_r}(G)$. If $U \subset V(G)$, we write $e(U)$ for $e(G[U])$, $k_H(U)$ for $k_H(G[U])$, and $k_r(U)$

for $k_r(G[U])$. A partition $V = \cup_{i=0}^k V_i$ is called *equitable*, if $|V_0| < k$, and $|V_1| = \dots = |V_k|$. A set of cardinality k is called a k -set.

In [8] Erdős raised the following problem (see also [4], p. 363).

Problem 1 *Let $c > 0$. Suppose $G = G(n, \lfloor cn^2 \rfloor)$ is such that*

$$e(W) \geq (c/4 + o(1))n^2$$

for every $W \subset V(G)$ with $|W| = \lfloor n/2 \rfloor$. Then, for every fixed r and sufficiently large n , the graph G contains K_r .

This problem was solved recently in [11], where the following more general result was proved.

Theorem 2 *For every $c > 0$ and $r \geq 3$, there exists $\beta = \beta(c, r) > 0$ such that, for every K_r -free graph $G = G(n, m)$ with $m \geq cn^2$, there exists a partition $V(G) = V_1 \cup V_2$ with $|V_1| = \lfloor n/2 \rfloor$, $|V_2| = \lceil n/2 \rceil$, and*

$$e(V_1, V_2) > (1/2 + \beta)m. \quad (1)$$

In fact, (1) is a lower bound on the MaxCut function for dense K_r -free graphs; note, that it differs significantly from those found in [1], [2] and [3]. We obtain a similar result about judicious partitions in Theorem 14.

Kohayakawa and Rödl [10] gave another solution to Problem 1; however, their method does not imply Theorem 2.

One of our goals in this note is to extend Theorem 2. We first prove the following basic result.

Theorem 3 *For all $\varepsilon > 0$ and $r \geq 2$, there exist $\xi = \xi(\varepsilon, r) > 0$ and $L = L(\varepsilon, r)$ such that, for every graph G of sufficiently large order n , the following assertion holds.*

If $k_r(G) < \xi n^r$, then there exists an equitable partition $V(G) = \cup_{i=0}^q V_i$ with $q < L$, and

$$e(V_i) < \varepsilon \binom{|V_i|}{2}$$

for every $i \in [q]$.

From this assertion we shall deduce that the conclusion of Theorem 2 remains essentially true under considerably weaker stipulations.

Theorem 4 *For all $c > 0$ and $r \geq 3$, there exist $\xi = \xi(c, r) > 0$ and $\beta = \beta(c, r) > 0$ such that, for n sufficiently large and every graph $G = G(n, m)$ with $m \geq cn^2$, the following assertion holds.*

If $k_r(G) < \xi n^r$, then there exists a partition $V(G) = V_1 \cup V_2$ with $|V_1| = \lfloor n/2 \rfloor$, $|V_2| = \lceil n/2 \rceil$, and

$$e(V_1, V_2) > (1/2 + \beta)m.$$

We deduce also a number of related results, in particular, the following analogue of Theorem 3.

Theorem 5 *For all $\varepsilon > 0$ and $r \geq 2$, there exist $\xi = \xi(\varepsilon, r) > 0$ and $L = L(\varepsilon, r)$ such that, for every graph H of order r , and every graph G of sufficiently large order n , the following assertion holds.*

If $k_H(G) < \xi n^r$, then there exists an equitable partition $V(G) = \cup_{i=0}^q V_i$ with $q < L$ such that

$$e(V_i) < \varepsilon \binom{|V_i|}{2} \quad \text{or} \quad e(V_i) > (1 - \varepsilon) \binom{|V_i|}{2}$$

for every $i \in [q]$.

Observe that, although Theorem 5 is a fairly general result, it does not imply Theorem 3 or its counterpart for independent r -sets.

Finally, we prove the following assertion that looks likely to be useful in Ramsey type applications; we shall investigate this topic in a forthcoming note.

Theorem 6 *For all $\varepsilon > 0$, $r \geq 2$ and $k \geq 2$, there exist $\delta = \delta(\varepsilon, r) > 0$, $\xi = \xi(\varepsilon, r) > 0$ and $L = L(\varepsilon, r, k)$ such that, for every graph G of sufficiently large order n , the following assertion holds.*

If $V(G) = \cup_{i=0}^k V_i$ is a δ -uniform partition such that

$$k_r(V_i) \leq \xi |V_i|^r \quad \text{or} \quad k_r(\overline{G[V_i]}) \leq \xi |V_i|^r$$

for every $i \in [k]$, then there exists an ε -uniform partition $V(G) = \cup_{i=0}^q W_i$ with $k \leq q \leq L$ such that

$$e(W_i) < \varepsilon \binom{|W_i|}{2} \quad \text{or} \quad e(W_i) > (1 - \varepsilon) \binom{|W_i|}{2}$$

for every $i \in [q]$.

The rest of the note is organized as follows. First we introduce some additional notation, then we prove Theorem 3 in Section 2, extend it in Section 3, and use it in Section 4 to prove Theorem 4. In Section 5 we prove Theorem 6 and, finally, in Section 6 we prove Theorem 5.

A few words about our proofs seem necessary. We apply continually Szemerédi's uniformity lemma (SUL) in a rather routine manner. However, for the reader's sake, we always provide the necessary details, despite repetitions.

1.1 Notation

Suppose G is a graph. For a vertex $u \in V(G)$, we write $\Gamma(u)$ for the set of vertices adjacent to u . If $A, B \subset V(G)$ are nonempty disjoint sets, we write $e(A, B)$ for the number of $A - B$ edges and set

$$d(A, B) = \frac{e(A, B)}{|A||B|}.$$

Given a partition $V = \cup_{i=0}^k V_i$, we occasionally call the sets V_1, \dots, V_k *clusters* of the partition.

For general notions and definitions related to Szemerédi's uniformity lemma (SUL), see, e.g. [9], or [5]. In our exposition we shall systematically replace “regularity” by “uniformity”, thus “ ε -uniform” will stand for “ ε -regular”.

Let $\varepsilon > 0$. A partition $V(G) = \cup_{i=0}^k V_i$ is called ε -uniform, if it is equitable, and at most εk^2 pairs (V_i, V_j) are not ε -uniform.

2 Proof of Theorem 3

In our proof of Theorem 3 and later we shall use SUL in the following form.

Theorem 7 (Szemerédi's Uniformity Lemma) *Let $l \geq 1$, $\varepsilon > 0$. There exists $M = M(\varepsilon, l)$ such that, for every graph G of sufficiently large order, there exists an ε -uniform partition $V(G) = \cup_{i=0}^k V_i$ with $l \leq k \leq M$.*

In addition, we need the following basic properties of ε -uniform pairs (see [9], Facts 1.4 and 1.5.)

Lemma 8 *Let $\varepsilon > 0$, $r \geq 1$, and (A, B) be an ε -uniform pair with $d(A, B) = d$. If $Y \subset B$ and $(d - \varepsilon)^{r-1} |Y| > \varepsilon |B|$, then there are at most $\varepsilon r |A|^r$ r -sets $R \subset A$ such that*

$$|(\cap_{u \in R} \Gamma(u)) \cap Y| \leq (d - \varepsilon)^r |Y|.$$

Lemma 9 *Let $0 < \varepsilon < \alpha$, and let (A, B) be an ε -uniform pair. If $A' \subset A$, $B' \subset B$ and $|A'| \geq \alpha |A|$, $|B'| \geq \alpha |B|$, then (A', B') is an ε' -uniform pair with $\varepsilon' = \max\{\varepsilon/\alpha, 2\varepsilon\}$.*

It is straightforward to deduce the following assertion from Lemma 8.

Lemma 10 *Let $r \geq 1$, $0 < 2\varepsilon^{1/r} < d \leq 1$, and let (A, B) be an ε -uniform pair with $d(A, B) = d$. There are at most $\varepsilon r |A|^r$ r -sets $R \subset A$ such that*

$$|(\cap_{u \in R} \Gamma(u)) \cap B| \leq \varepsilon |B|.$$

The following simple lemma will play a crucial role in our proofs.

Lemma 11 (Scooping Lemma) *Let $\varepsilon > 0$, and let s be integer with $0 < s \leq \varepsilon n$. For every graph G of order n , if $e(G) \leq \varepsilon^3 \binom{n}{2}$, then there exists a partition $V(G) = \cup_{i=0}^k V_i$ such that $|V_0| \leq \lceil \varepsilon n \rceil$, and*

$$|V_i| = s, \quad e(V_i) < \varepsilon \binom{s}{2}$$

for every $i \in [k]$.

Proof Select a sequence of sets V_1, \dots, V_k as follows: select V_1 by

$$e(V_1) = \min \{e(U) : U \subset V(G), |U| = s\};$$

having selected V_1, \dots, V_i , if $n - is \leq \lceil \varepsilon n \rceil$ stop the sequence, else select V_{i+1} by

$$e(V_{i+1}) = \min \{e(U) : U \subset V(G) \setminus (\cup_{j=1}^i V_j), |U| = s\}.$$

Let V_k be the last selected set; set $V_0 = V(G) \setminus (\cup_{i=1}^k V_i)$. The stop condition implies $|V_0| \leq \lceil \varepsilon n \rceil$. For every $i \in [k]$, the way we choose V_i implies

$$\begin{aligned} e(V_i) &\leq \frac{e(G)}{\binom{n-(i-1)s}{2}} \binom{s}{2} \leq \frac{\varepsilon^3 n(n-1)}{(n-(i-1)s)(n-(i-1)s-1)} \binom{s}{2} \\ &< \frac{\varepsilon^3 n(n-1)}{(\lceil \varepsilon n \rceil + 1) \lceil \varepsilon n \rceil} \binom{s}{2} < \varepsilon \binom{s}{2}, \end{aligned}$$

so the partition $V(G) = \cup_{i=0}^k V_i$ has the required properties. \square

Proof of Theorem 3 Setting $q = L(\varepsilon, 2) = 1$, $\xi(\varepsilon, 2) = \varepsilon$, the theorem holds trivially for $r = 2$. To prove it for $r > 2$ we apply induction on r - assuming it holds for r , we shall prove it for $r + 1$.

Observe that it suffices to find $\xi = \xi(\varepsilon, r + 1) > 0$ and $L = L(\varepsilon, r + 1)$ such that, if G is a graph of sufficiently large order n , and $k_{r+1}(G) < \xi n^{r+1}$, then there exists a partition $V(G) = \cup_{i=0}^q W_i$ such that:

- (i) $q \leq L$;
- (ii) $|W_0| < 6\varepsilon n$, $|W_1| = \dots = |W_q|$;
- (iii) for every $i \in [q]$, $e(W_i) < \varepsilon \binom{|W_i|}{2}$.

Indeed, distributing evenly among the sets W_1, \dots, W_q as many as possible of the vertices of W_0 , we obtain a partition $V(G) = \cup_{i=0}^q V_i$ with $|V_0| < q$, and

$$|V_i| = \lfloor n/q \rfloor, \quad e(V_i) < 2\varepsilon \binom{|V_i|}{2},$$

for every $i \in [q]$, as required.

For convenience we shall outline first our proof. For δ appropriately small, applying SUL, we find a δ -uniform partition $V(G) = \cup_{i=0}^k V_i$. Note that, if $k_r(V_i)$ is proportional to n^r , and V_i is incident to a substantially dense δ -uniform pair, then, by Lemma 10, there are substantially many $(r + 1)$ -cliques in G . Therefore, for every V_i , either $k_r(V_i)$ is small or V_i is essentially isolated.

Let V'' be the union of the essentially isolated clusters; set $V' = V \setminus (V'' \cup V_0)$.

1 Partitioning of V'

By the induction hypothesis, we partition each nonisolated V_i into a bounded number of sparse sets Y_{ij} and a small exceptional set; the exceptional sets are collected in X' . Note that, although the sets Y_{ij} are sparse, they are not good for our purposes, for their cardinality may vary with i . To overcome this obstacle, we first select a sufficiently small integer s proportional to n . Then, by the Scooping Lemma, we partition each of the sets Y_{ij} into sparse sets of cardinality exactly s and a small exceptional set; the exceptional sets are added to X' .

2 Partitioning of V''

We partition V'' into sparse sets of size s and a small exceptional set X_0 . If $|V''|$ is small, we set $X_0 = V''$, and complete the partition. If $|V''|$ is substantial, then $G[V'']$ must be sparse, for it consists of essentially isolated clusters. Applying the Scooping Lemma to $G[V'']$, we partition V'' into sparse sets of cardinality s and a small exceptional set X_0 .

Let W_1, \dots, W_q be the sets of cardinality s obtained during the partitioning of V' and V'' . Set $W_0 = V_0 \cup X_0 \cup X'$; the choice of δ implies $|W_0| < 6\varepsilon n$, so the partition $V(G) = \cup_{i=0}^q W_i$ satisfies (i)-(iii).

Let us now give the details. Assume ε sufficiently small and set

$$l = \max \left\{ \left\lceil \frac{1}{\varepsilon^5} \right\rceil, \frac{1}{4\varepsilon L(\varepsilon^3, r)} \right\}, \quad (2)$$

$$\delta = \min \left\{ \frac{\xi(\varepsilon^3, r)}{r+1}, \frac{\varepsilon^{5r}}{16^r} \right\}. \quad (3)$$

$$L = L(\varepsilon, r+1) = \frac{8M(\delta, l)L(\varepsilon^3, r)}{\varepsilon}. \quad (4)$$

$$\xi = \xi(\varepsilon, r+1) = \frac{\delta^2}{(2M(\delta, l))^{r+1}} \quad (5)$$

Let G be a graph of sufficiently large order n , and let $k_{r+1}(G) < \xi n^{r+1}$. Applying SUL, we find a δ -uniform partition $V(G) = \cup_{i=0}^k V_i$ with $l \leq k \leq M(\delta, l)$. Set $t = |V_1|$ and observe that

$$\frac{n}{2k} \leq (1-\delta) \frac{n}{k} < t \leq \frac{n}{k}. \quad (6)$$

Assume that there exist a cluster V_i with $k_r(V_i) > \xi(\varepsilon^3, r)t^r$, and a δ -uniform pair (V_i, V_j) with

$$d(V_i, V_j) > 2\delta^{1/r}.$$

Applying Lemma 10 with $A = V_i$ and $B = V_j$, we find that there are at least

$$\xi(\varepsilon^3, r)t^r - \delta r t^r \geq \delta t^r$$

r -cliques $R \subset V_i$ such that

$$|(\cap_{u \in R} \Gamma(u)) \cap V_j| > \delta t.$$

Hence, there are at least $\delta^2 t^{r+1} (r+1)$ -cliques inducing an r -clique in V_i and a vertex in V_j . Therefore, from (6) and (5), we find that

$$\begin{aligned} k_{r+1}(G) &\geq \delta^2 t^{r+1} > \delta^2 \left(\frac{1-\delta}{k} \right)^{r+1} n^{r+1} \\ &> \frac{\delta^2}{(2M(\delta, l))^{r+1}} n^{r+1} = \xi(\varepsilon, r+1) n^{r+1}, \end{aligned}$$

a contradiction. Therefore, if $k_r(V_i) \geq \xi(\varepsilon^3, r) t^r$, then every ε -uniform pair (V_i, V_j) satisfies $d(V_i, V_j) \leq 2\delta^{1/r}$. Let

$$I' = \{i : i \in [k], k_r(V_i) \leq \xi(\varepsilon^3, r) t^r\}, \quad I'' = [k] \setminus I'.$$

First we shall partition $V' = \cup_{i \in I'} V_i$. Set

$$s = \left\lfloor \frac{\varepsilon n}{4kL(\varepsilon^3, r)} \right\rfloor \quad (7)$$

and observe that

$$\frac{n}{s} < \frac{8kL(\varepsilon^3, r)}{\varepsilon} \leq \frac{8M(\delta, l)L(\varepsilon^3, r)}{\varepsilon} = L(\varepsilon, r+1). \quad (8)$$

For every $i \in I'$, by the induction hypothesis, we find an equitable partition $V_i = \cup_{j=0}^{m_i} Y_{ij}$ with $|Y_{i0}| < m_i \leq L(\varepsilon^3, r)$, and

$$e(Y_{ij}) \leq \varepsilon^3 \binom{|Y_{ij}|}{2}$$

for every $j \in [m_i]$. Also, for every $i \in I'$ and $j \in [m_i]$, (7) and (6) imply

$$s \leq \frac{\varepsilon n}{4kL(\varepsilon^3, r)} < \varepsilon \frac{t}{2L(\varepsilon^3, r)} \leq \varepsilon \frac{t}{2m_i} \leq \varepsilon \left\lfloor \frac{t}{m_i} \right\rfloor = \varepsilon |Y_{ij}|.$$

Hence, we apply the Scooping Lemma to the graph $G[Y_{ij}]$, and find a partition $Y_{ij} = \cup_{q=0}^{p_{ij}} W_{ijq}$ with $|W_{ij0}| \leq \lceil \varepsilon |Y_{ij}| \rceil$ such that

$$|W_{ijq}| = s \quad \text{and} \quad e(W_{ijq}) < \varepsilon \binom{s}{2}$$

for every $q \in [p_{ij}]$. Setting $X' = (\cup_{i \in I'} Y_{i0}) \cup (\cup_{i \in I'} \cup_{j=1}^{m_i} W_{ij0})$, we obtain

$$\begin{aligned} |X'| &= |(\cup_{i \in I'} Y_{i0})| + |(\cup_{i \in I'} \cup_{j=1}^{m_i} W_{ij0})| < \sum_{i \in I'} m_i + 2\varepsilon \sum_{i \in I'} \sum_{j=1}^{m_i} |Y_{ij}| \\ &< kL(\varepsilon^3, r) + 2\varepsilon \sum_{i \in I'} m_i \left\lfloor \frac{t}{m_i} \right\rfloor \leq kL(\varepsilon^3, r) + 2\varepsilon n < 3\varepsilon n. \end{aligned} \quad (9)$$

Denote by h the number of the sets W_{ijq} ($i \in I'$, $j \in [m_i]$, $q \in [p_{ij}]$), and renumber them sequentially from 1 to h . So far we have a partition $V' = X' \cup (\cup_{i=1}^h W_i)$ with $|X'| < 3\varepsilon n$ such that

$$|W_i| = s, \quad e(W_i) < \varepsilon \binom{s}{2}$$

for every $i \in [h]$.

Next we shall partition the set $V'' = \cup_{i \in I''} V_i$. We may assume that $|V''| \geq \varepsilon n$, else, setting $W_0 = V_0 \cup X' \cup V''$, from (9), we have $W_0 < 5\varepsilon n$, and, in view of (8), the proof is completed. Obviously,

$$e(V'') = \sum_{i \in I''} e(V_i) + \sum_{i, j \in I'', i < j} e(V_i, V_j) \leq k \binom{t}{2} + e_1 + e_2, \quad (10)$$

where,

$$\begin{aligned} e_1 &= \sum_{i, j \in I'', i < j} \{e(V_i, V_j) : (V_i, V_i) \text{ is } \delta\text{-uniform}\}, \\ e_2 &= \sum_{i, j \in I'', i < j} \{e(V_i, V_j) : (V_i, V_i) \text{ is not } \delta\text{-uniform}\} \end{aligned}$$

Since at most δk^2 pairs (V_i, V_i) , $(1 \leq i < j \leq k)$, are not δ -uniform, it follows that

$$e_2 \leq \delta k^2 t^2 \leq \delta n^2. \quad (11)$$

Recall that if $i, j \in I''$, $i < j$, and the pair (V_i, V_j) is δ -uniform, then $d(V_i, V_j) < 2\delta^{1/r}$. Therefore,

$$e_1 \leq \binom{k}{2} 2\delta^{1/r} t^2 \leq \delta^{1/r} k^2 t^2 \leq \delta^{1/r} n^2.$$

Hence, (10), (11), (6), and (2) imply

$$\begin{aligned} e(V'') &\leq \frac{kt^2}{2} + \delta n^2 + \delta^{1/r} n^2 \leq \left(\frac{1}{2k} + 2\delta^{1/r}\right) n^2 \leq \left(\frac{1}{2l} + 2\delta^{1/r}\right) n^2 \\ &\leq \left(\frac{\varepsilon^5}{8} + \frac{\varepsilon^5}{8}\right) n^2 \leq \frac{\varepsilon^3}{4} |V''|^2 < \varepsilon^3 \binom{|V''|}{2}. \end{aligned}$$

On the other hand, (7) and (2) imply

$$s \leq \frac{\varepsilon n}{4kL(\varepsilon^3, r)} \leq \frac{\varepsilon n}{4lL(\varepsilon^3, r)} \leq \varepsilon^2 n < \varepsilon |V''|. \quad (12)$$

Hence, we apply the Scooping Lemma to the graph $G[V'']$, and find a partition $V'' = \cup_{i=0}^g X_i$ such that $|X_0| < \lceil \varepsilon |V''| \rceil$, and $|X_1| = \dots = |X_g| = s$. Set

$$\begin{aligned} q &= h + g, \\ W_0 &= V_0 \cup X_0 \cup X', \\ W_{h+i} &= X_i, \quad i \in [g]. \end{aligned}$$

From (9) and (12) it follows

$$|W_0| = |V_0| + |X_0| + |X'| < M(\delta, l) + \lceil \varepsilon |V''| \rceil + 3\varepsilon n < 6\varepsilon n.$$

Finally, (8) implies

$$q = h + g \leq \frac{n}{s} < L(\varepsilon, r + 1),$$

completing the proof. \square

3 Extensions of Theorem 3

Generally speaking, Theorem 3 states that, if certain conditions about a graph are met, then its vertices can be partitioned in a specific way. It turns out that, in addition, the partition may be selected to be ε -uniform. This is the topic of the following two theorems.

Theorem 12 *For all $\varepsilon > 0$, $r \geq 2$ and $k \geq 2$, there exist $\rho = \rho(\varepsilon, r, k) > 0$ and $K = K(\varepsilon, r, k)$ such that, for every graph G of sufficiently large order n , the following assertion holds*

If $k_r(G) \leq \rho n^r$, then there exists an ε -uniform partition $V(G) = \cup_{i=0}^q V_i$ with $k \leq q \leq K$, and

$$e(V_i) \leq \varepsilon \binom{|V_i|}{2}$$

for every $i \in [q]$.

Proof Our proof is essentially the same as the proof of Theorem 3.

Suppose $M(\varepsilon, l)$ is as defined in SUL, and $\xi(\varepsilon, r)$, $L(\varepsilon, r)$ are as defined in Theorem 3. Assume ε sufficiently small and set

$$\delta = \min \left\{ \frac{\varepsilon^2}{8L(\varepsilon^3, r)}, \frac{\varepsilon}{4} \right\} \quad (13)$$

$$l = \max \left\{ k, \frac{2}{\varepsilon} \right\}. \quad (14)$$

$$K = K(\varepsilon, r, k) = \frac{8M(\delta, l)L(\varepsilon^3, r)}{\varepsilon} \quad (15)$$

$$\rho = \rho(\varepsilon, r, k) = \frac{\xi(\varepsilon^3, r)}{(2M(\delta, l))^r}. \quad (16)$$

Let G be a graph of sufficiently large order n , and let $k_r(G) \leq \rho n^r$. It suffices to find a partition $V(G) = \cup_{i=0}^q W_i$ such that:

- (i) $k \leq q \leq K$;
- (ii) $|W_0| < 3\varepsilon n$, $|W_1| = \dots = |W_q|$;
- (iii) for every $i \in [q]$, $e(W_i) < \varepsilon |W_i|^2$;
- (iv) at most εq^2 pairs are not ε -uniform.

Applying SUL, we find a δ -uniform partition $V(G) = \cup_{i=0}^p V_i$ with $l \leq p \leq M(\delta, l)$. Set $t = |V_1|$ and observe that

$$\frac{n}{2p} \leq (1 - \delta) \frac{n}{p} < t \leq \frac{n}{p}. \quad (17)$$

For every $i \in [p]$, we have

$$k_r(V_i) \leq k_r(G) < \rho n^r = \frac{\xi(\varepsilon^3, r)}{(2M(\delta, l))^r} n^r \leq \xi(\varepsilon^3, r) \left(\frac{n}{2p} \right)^r \leq \xi(\varepsilon^3, r) t^r.$$

Hence, for every $i \in [p]$, we apply Theorem 3, and find an equitable partition $V_i = \cup_{j=0}^{m_i} Y_{ij}$ with $m_i \leq L(\varepsilon^3, r)$, and

$$e(Y_{ij}) \leq \varepsilon^3 \binom{|Y_{ij}|}{2} \quad (18)$$

for every $j \in [m_i]$. Set

$$s = \left\lfloor \frac{\varepsilon n}{4pL(\varepsilon^3, r)} \right\rfloor \quad (19)$$

and observe that

$$\frac{n}{s} < \frac{8pL(\varepsilon^3, r)}{\varepsilon} < \frac{8M(\delta, l)L(\varepsilon^3, r)}{\varepsilon} = K(\varepsilon, r, k). \quad (20)$$

Also, for every $i \in [p]$ and $j \in [m_i]$, (19) and (17) imply

$$s \leq \frac{\varepsilon n}{4pL(\varepsilon^3, r)} < \varepsilon \frac{t}{2L(\varepsilon^3, r)} \leq \varepsilon \frac{t}{2m_i} < \varepsilon \left\lfloor \frac{t}{m_i} \right\rfloor = \varepsilon |Y_{ij}|.$$

Hence, for every $i \in [p]$ and $j \in [m_i]$, in view of (18), we apply the Scooping Lemma to the graph $G[Y_{ij}]$, and find a partition $Y_{ij} = \cup_{h=0}^{p_{ij}} W_{ijh}$ with $|W_{ij0}| \leq \lceil \varepsilon |Y_{ij}| \rceil$ such that $|W_{ijh}| = s$, and $e(W_{ijh}) < \varepsilon \binom{s}{2}$ for every $h \in [p_{ij}]$. Setting

$$W_0 = V_0 \cup (\cup_{i \in I} Y_{i0}) \cup (\cup_{i \in I} \cup_{j=1}^{m_i} W_{ij0}),$$

we obtain

$$\begin{aligned} |W_0| &= |V_0| + |(\cup_{i \in I} Y_{i0})| + |(\cup_{i \in I} \cup_{j=1}^{m_i} W_{ij0})| < |V_0| + \sum_{i=1}^p m_i + \sum_{i=1}^p \sum_{j=1}^{m_i} \lceil \varepsilon |Y_{ij}| \rceil \\ &< M(\delta, l) + pL(\varepsilon^3, r) + 2\varepsilon \sum_{i=1}^p m_i \left\lfloor \frac{t}{m_i} \right\rfloor < 3\varepsilon n. \end{aligned} \quad (21)$$

Denote by q the number of the sets W_{ijh} ($i \in I'$, $j \in [m_i]$, $h \in [p_{ij}]$), and renumber them sequentially from 1 to q . Clearly, from (21) and (20), we have

$$\frac{(1-3\varepsilon)n}{s} \leq q \leq \frac{n}{s} \leq K(\varepsilon, r, k). \quad (22)$$

Let us check that the partition $V(G) = \cup_{i=0}^q W_i$ satisfies (i)-(iv). For every $i \in [p]$, the cluster V_i contains at least one W_j ($j \in [q]$), so (i) holds. Observe that $|W_0| < 3\varepsilon n$, and

$$|W_i| = s, \quad e(W_i) < \varepsilon \binom{|W_i|}{2}$$

for every $i \in [q]$, so (ii) and (iii) also hold. To complete the proof, it remains to check (iv). Suppose $W_a \subset V_i$, $W_b \subset V_j$. If the pair (V_i, V_j) is δ -uniform, then,

$$|W_a| = |W_b| = s \geq \frac{\varepsilon n}{8kL(\varepsilon^3, r)} \geq \frac{\varepsilon}{8L(\varepsilon^3, r)} t.$$

Since (13) implies

$$\varepsilon = \max \left\{ \frac{8L(\varepsilon^3, r)}{\varepsilon} \delta, 2\delta \right\},$$

from Lemma 9, it follows that the pair (W_a, W_b) is ε -uniform. Therefore, if the pair (W_a, W_b) is not ε -uniform, then either $i = j$ or the pair (V_i, V_j) is not δ -uniform. For every $i \in [p]$, V_i contains at most $\lfloor t/s \rfloor$ sets W_a , so the number of the pairs (W_a, W_b) that are not ε -uniform is at most

$$p \binom{\lfloor t/s \rfloor}{2} + \delta p^2 \lfloor t/s \rfloor^2 < \left(\frac{1}{2p} + \delta \right) \left(\frac{pt}{s} \right)^2 \leq \left(\frac{1}{2l} + \delta \right) \left(\frac{n}{s} \right)^2.$$

From (13), (??) and (22) we find that

$$\left(\frac{1}{2l} + \delta \right) \left(\frac{n}{s} \right)^2 \leq \frac{\varepsilon}{2} \left(\frac{n}{s} \right)^2 < \varepsilon \left(\frac{(1-3\varepsilon)n}{s} \right)^2 \leq \varepsilon q^2,$$

completing the proof. \square

Applying routine argument, we obtain the following corollary.

Theorem 13 *For all $\varepsilon > 0$, $r \geq 2$ and $k \geq 2$, there exist $\rho = \rho(\varepsilon, r, k) > 0$ and $K = K(\varepsilon, r, k)$ such that, for every graph G of sufficiently large order n , the following assertion holds.*

If $k_r(G) \leq \rho n^r$, then there exists a partition $V(G) = \cup_{i=1}^q V_i$ with $k \leq q \leq K$ such that

$$\lfloor n/q \rfloor \leq |V_i| \leq \lceil n/q \rceil, \quad e(V_i) \leq \varepsilon \binom{|V_i|}{2}$$

for every $i \in [q]$, and at most εq^2 pairs (W_i, W_j) are not ε -uniform.

4 Bipartitions of low density

In this section we shall deduce Theorem 4. We first state and prove a preliminary result of its own interest. In fact, this is a particular result on judicious bipartitions of dense graphs with moderately many r -cliques; it significantly differs from known general results as in [6], [7] and [2].

Theorem 14 *For all $r \geq 3$, $c > 0$ and $\varepsilon > 0$, there exist $\xi = \xi(\varepsilon, c, r) > 0$ and $\beta = \beta(\varepsilon, c, r) > 0$ such that, for every graph $G = G(n, \lfloor cn^2 \rfloor)$ of sufficiently large order n , the following assertion holds.*

If $k_r(G) < \xi n^r$, then there exists a partition $V(G) = V_1 \cup V_2$ such that

$$e(V_1) < \varepsilon |V_1|^2 \quad \text{and} \quad e(V_2) < (c - \beta) |V_2|^2.$$

Proof Let $\xi_1(\varepsilon, r)$ and $L_1(\varepsilon, r)$ correspond to $\xi(\varepsilon, r)$ and $L(\varepsilon, r)$ as defined in Theorem 3. Set

$$\begin{aligned}\sigma &= \min\{c/4, \varepsilon\} \\ \xi &= \xi(\varepsilon, c, r) = \xi_1(\sigma, r) \\ L &= L(\varepsilon, c, r) = L_1(\sigma, r) \\ \beta &= \beta(\varepsilon, c, r) = \frac{c - 2\sigma}{L^2}\end{aligned}\tag{23}$$

Let the graph $G = G(n, \lfloor cn^2 \rfloor)$ be with $k_r(G) < \xi n^r$. If n is sufficiently large, we apply Theorem 12, and find a σ -uniform partition $V(G) = \cup_{i=0}^k W_i$ with $k \leq L$ such that $|W_i| = \lfloor n/k \rfloor$ and $e(W_i) \leq \sigma \lfloor n/k \rfloor^2$ for every $i \in [k]$. Set $V' = V \setminus W_0$. We may and shall assume that the cluster W_1 satisfies

$$e(W_1, V' \setminus W_1) = \max_{i \in [k]} \{e(W_i, V' \setminus W_i)\}.$$

We shall prove that the partition $V(G) = V_1 \cup V_2$, defined with $V_1 = W_1$, $V_2 = V \setminus W_1$, satisfies the requirements. Indeed, we immediately have

$$e(V_1) < \sigma |V_1|^2 \leq \varepsilon |V_1|^2.$$

Therefore, all we have to prove is that, for n sufficiently large, $e(V_2) < (c - \beta) |V_2|^2$, that is to say

$$e(V \setminus W_1) < (c - \beta) \left(n - \left\lfloor \frac{n}{k} \right\rfloor\right)^2.\tag{24}$$

We have,

$$e(V') = e(V) - e(W_0, V') - e(W_0) \geq e(V) - |W_0|n > e(V) - kn.\tag{25}$$

On the other hand,

$$\begin{aligned}e(V') &= \sum_{i=1}^k e(W_i) + \frac{1}{2} \sum_{i=1}^k e(W_i, V' \setminus W_i) \leq \sum_{i=1}^k e(W_i) + \frac{k}{2} e(W_1, V' \setminus W_1) \\ &\leq \sigma k \left\lfloor \frac{n}{k} \right\rfloor^2 + \frac{k}{2} e(W_1, V' \setminus W_1).\end{aligned}$$

Therefore,

$$e(W_1, V' \setminus W_1) \geq \frac{2e(V')}{k} - 2\sigma \left\lfloor \frac{n}{k} \right\rfloor^2.$$

This, together with (25), implies

$$\begin{aligned}e(V \setminus W_1) &= e(V) - e(W_1) - e(W_1, V \setminus W_1) \leq e(V) - e(W_1, V' \setminus W_1) \\ &\leq e(V) - \frac{2e(V')}{k} + 2\sigma \left\lfloor \frac{n}{k} \right\rfloor^2 \leq \frac{k-2}{k} e(V) + kn + 2\sigma \left\lfloor \frac{n}{k} \right\rfloor^2.\end{aligned}$$

Hence, in view of $e(V) = \lfloor cn^2 \rfloor$, we deduce

$$e(V \setminus W_1) \leq \frac{k-2}{k} \lfloor cn^2 \rfloor + kn + 2\sigma \left\lfloor \frac{n}{k} \right\rfloor^2. \quad (26)$$

Assume that there are arbitrary large values of n for which (24) is false, thus

$$e(V \setminus W_1) \geq (c - \beta) \left(n - \left\lfloor \frac{n}{k} \right\rfloor \right)^2$$

holds. Hence, in view of (26), we find that

$$(c - \beta) \left(n - \left\lfloor \frac{n}{k} \right\rfloor \right)^2 \leq \frac{k-2}{k} \lfloor cn^2 \rfloor + kn + 2\sigma \left\lfloor \frac{n}{k} \right\rfloor^2.$$

Dividing both sides by n^2 and taking the limit, we deduce

$$(c - \beta) \left(1 - \frac{1}{k} \right)^2 \leq \frac{k-2}{k} c + \frac{2\sigma}{k^2},$$

and hence,

$$c - 2\sigma \leq \beta(k-1)^2 < \beta L^2,$$

a contradiction with (23). The proof is completed. \square

In [11], for every graph $G = G(n, m)$, the function

$$\Phi(G, k) = \min_{U \subset V(G), |U|=k} \left\{ \frac{e(U)}{k} + \frac{e(V \setminus U)}{n-k} - \frac{m}{n} \right\}$$

is introduced, and, it is shown that, if $1 \leq k \leq \lfloor n/2 \rfloor$, then

$$\Phi\left(G, \left\lfloor \frac{n}{2} \right\rfloor\right) \leq \frac{k}{n-k} \Phi(G, k).$$

This inequality, together with Theorem 14, easily implies Theorem 2.

Note that $k_r(\overline{G})$ is exactly the number of independent r -sets in G . Restating the Scooping Lemma, and Theorems 3, 4, 12 and 13, for the complementary graph, we obtain equivalent assertions for graphs with few independent r -sets; for example, the following theorem is equivalent to Theorem 4.

Theorem 15 *For all $c > 0$ and $r \geq 3$, there exist $\xi = \xi(c, r) > 0$ and $\beta = \beta(c, r) > 0$ such that, for n sufficiently large, and every graph $G = G(n, m)$ with $m \geq cn^2$, the following assertion holds.*

If $k_r(\overline{G}) < \xi n^r$, then there exists a partition $V(G) = V_1 \cup V_2$ with $|V_1| = \lfloor n/2 \rfloor$, $|V_2| = \lceil n/2 \rceil$, and

$$e(V_1, V_2) < (1/2 - \beta)m.$$

5 Refining partitions

This section contains a proof of Theorem 6.

Proof of Theorem 6 We follow essentially the proof of Theorem 12.

Assume $\xi(\varepsilon, r)$ and $L(\varepsilon, r)$ as defined in Theorem 3; assume ε sufficiently small and set

$$\delta = \delta(\varepsilon, r) = \min \left\{ \frac{\varepsilon^2}{8L(\varepsilon^3, r)}, \frac{\varepsilon}{8} \right\} \quad (27)$$

$$\rho = \rho(\varepsilon, r) = \xi(\varepsilon^3, r) \quad (28)$$

$$K = K(\varepsilon, r, k) = \frac{8kL(\varepsilon^3, r)}{\varepsilon} \quad (29)$$

Let G be a graph of sufficiently large order n , and let $V(G) = \cup_{i=0}^k V_i$ be a δ -uniform partition such that

$$k_r(V_i) \leq \rho \lfloor n/k \rfloor^r \quad \text{or} \quad k_r(\overline{G[V_i]}) \leq \rho \lfloor n/k \rfloor^r \quad (30)$$

for every $i \in [k]$. We shall also assume that

$$k > \frac{2}{\varepsilon}, \quad (31)$$

as, changing δ , ρ and K appropriately, we may refine the partition $V(G) = \cup_{i=0}^k V_i$ so that (30) and (31) hold. To prove the theorem, it suffices to find a partition $V(G) = \cup_{i=0}^q W_i$ such that:

- (i) $k \leq q \leq K$;
- (ii) $|W_0| < 3\varepsilon n$, $|W_1| = \dots = |W_q|$;
- (iii) for every $i \in [q]$, $e(W_i) < \varepsilon \binom{|W_i|}{2}$ or $e(W_i) < (1 - \varepsilon) \binom{|W_i|}{2}$;
- (iv) at most εq^2 pairs are not ε -uniform.

Set $t = |V_1|$ and observe that

$$\frac{n}{2k} \leq (1 - \delta) \frac{n}{k} < t \leq \frac{n}{k}. \quad (32)$$

From (28) and (30) it follows

$$k_r(V_i) \leq \xi(\varepsilon^3, r) t^r \quad \text{or} \quad k_r(\overline{G[V_i]}) \leq \xi(\varepsilon^3, r) t^r$$

for every $i \in [k]$. Hence, for every $i \in [k]$, we apply Theorem 3 to the graph $G(V_i)$ or to its complement, and find an equitable partition $V_i = \cup_{j=0}^{m_i} Y_{ij}$ with $m_i \leq L(\varepsilon^3, r)$ such that

$$e(Y_{ij}) \leq \varepsilon^3 \binom{|Y_{ij}|}{2} \quad \text{or} \quad e(Y_{ij}) \geq (1 - \varepsilon^3) \binom{|Y_{ij}|}{2}$$

for every $j \in [m_i]$. Set

$$s = \left\lfloor \frac{\varepsilon n}{4kL(\varepsilon^3, r)} \right\rfloor \quad (33)$$

and observe that

$$\frac{n}{s} < \frac{8kL(\varepsilon^3, r)}{\varepsilon} = K(\varepsilon, r, k). \quad (34)$$

Also, note that, for every $i \in [k]$, $j \in [m_i]$, (32) and (33) imply

$$s \leq \frac{\varepsilon n}{4kL(\varepsilon^3, r)} < \varepsilon \frac{t}{2K(\varepsilon^3, r, l)} \leq \varepsilon \frac{t}{2m_i} \leq \varepsilon \left\lfloor \frac{t}{m_i} \right\rfloor = \varepsilon |Y_{ij}|.$$

Hence, for every $i \in [k]$, $j \in [m_i]$, we apply the Scooping Lemma to the graph $G[Y_{ij}]$ or its complement, and find a partition $Y_{ij} = \cup_{h=0}^{p_{ij}} W_{ijh}$ with $|W_{ij0}| \leq \lceil \varepsilon |Y_{ij}| \rceil$ such that

$$|W_{ijh}| = s, \quad \text{and} \quad e(W_{ijh}) < \varepsilon \binom{s}{2} \quad \text{or} \quad e(W_{ijh}) > (1 - \varepsilon) \binom{s}{2}$$

for every $h \in [p_{ij}]$. Setting

$$W_0 = V_0 \cup \left(\cup_{i=1}^k Y_{i0} \right) \cup \left(\cup_{i=1}^k \cup_{j=1}^{m_i} W_{ij0} \right),$$

we obtain

$$\begin{aligned} |W_0| &= |V_0| + |(\cup_{i \in I} Y_{i0})| + |(\cup_{i \in I} \cup_{j=1}^{m_i} W_{ij0})| < |V_0| + \sum_{i=1}^k m_i + \sum_{i=1}^k \sum_{j=1}^{m_i} \lceil \varepsilon |Y_{ij}| \rceil \\ &< k + kL(\varepsilon^3, r) + 2\varepsilon \sum_{i=1}^k m_i \left\lfloor \frac{t}{m_i} \right\rfloor < 3\varepsilon n. \end{aligned} \quad (35)$$

Denote by q be the number of the sets W_{ijh} ($i \in I'$, $j \in [m_i]$, $h \in [p_{ij}]$), and renumber them sequentially from 1 to q . Clearly, from (35) and (34), we have

$$\frac{(1 - 3\varepsilon)n}{s} \leq q \leq \frac{n}{s} \leq K(\varepsilon, r, k). \quad (36)$$

Let us check that the partition $V(G) = \cup_{i=0}^q W_i$ satisfies (i)-(iv). For every $i \in [k]$, the cluster V_i contains at least one W_j ($j \in [q]$), so (i) holds. Observe that $|W_0| < 3\varepsilon n$, and

$$|W_i| = s, \quad e(W_i) < \varepsilon \binom{s}{2}$$

for every $i \in [q]$, so (ii) and (iii) also hold. To complete the proof, it remains to check (iv). Suppose $W_a \subset V_i$, $W_b \subset V_j$. If the pair (V_i, V_j) is δ -uniform then

$$|W_a| = |W_b| = s \geq \frac{\varepsilon n}{8kL(\varepsilon^3, r)} \geq \frac{\varepsilon}{8L(\varepsilon^3, r)} t.$$

Since, from (27), we have

$$\varepsilon = \max \left\{ \frac{8L(\varepsilon^3, r)}{\varepsilon} \delta, 2\delta \right\},$$

Lemma 9 implies that the pair (W_a, W_b) is ε -uniform. Therefore, if the pair (W_a, W_b) is not ε -uniform, then either $i = j$, or the pair (V_i, V_j) is not δ -uniform. For every $i \in [k]$, V_i contains at most $\lfloor t/s \rfloor$ sets W_a , so the number of the pairs (W_a, W_b) that are not ε -uniform is at most

$$k \binom{\lfloor t/s \rfloor}{2} + \delta k^2 \lfloor t/s \rfloor^2 < \left(\frac{1}{2k} + \delta \right) \left(\frac{kt}{s} \right)^2 \leq \left(\frac{1}{2k} + \delta \right) \left(\frac{n}{s} \right)^2.$$

From (27) and (31) we find that

$$\left(\frac{1}{2k} + \delta \right) \left(\frac{n}{s} \right)^2 \leq \frac{\varepsilon}{2} \left(\frac{n}{s} \right)^2 < \varepsilon \left(\frac{(1-3\varepsilon)n}{s} \right)^2 \leq \varepsilon q^2,$$

completing the proof. \square

6 Induced subgraphs

In this section we shall prove Theorem 5. We start by a simple partitioning lemma.

Lemma 16 *For all $\varepsilon > 0$ and $b \geq 2$, there exist $\gamma = \gamma(\varepsilon, b)$ and $n(\varepsilon, b)$ such that, for $n > n(\varepsilon, b)$, if the edges of K_n are colored in red, blue and green, then the following assertion holds.*

If there are fewer than γn^2 green edges, then there exists a partition $V(K_n) = \cup_{i=0}^q V_i$ such that $|V_0| < \varepsilon n$, $|V_1| = \dots = |V_q| = b$, and V_i spans either a red or a blue b -clique for every $i \in [n]$.

Proof Ramsey's theorem implies that, for every b , there exists $r = r(b)$ such that, if $n > r$ and the edges of K_n are colored in two colors, then there exists a monochromatic K_b .

We shall assume $\varepsilon < 1$, else there is nothing to prove. Set

$$\gamma = \gamma(\varepsilon, b) = \frac{\varepsilon^2}{4r}.$$

Suppose $n > 2r/\varepsilon^2$ and let the edges of K_n be colored in red, blue and green, so that there are fewer than γn^2 green edges. Therefore, there are at least

$$\binom{n}{2} - \gamma n^2 = \frac{n^2}{2} - \frac{n}{2} - \frac{n^2}{4r} > \frac{(r-1)n^2}{2r}$$

red or blue edges. Hence, by Turán's theorem, there is a set U of cardinality $r+1$ inducing only red or blue edges. By the choice of r , U induces a red or a blue b -clique; select one and denote its vertex set by V_1 . Proceed selecting sets V_2, \dots, V_q as follows: having selected V_1, \dots, V_i , if

$$\binom{n-bi}{2} - \frac{\varepsilon^2}{4r} n^2 < \frac{(r-1)(n-bi)^2}{2r}$$

stop the sequence, else, by Turán's theorem, find a set U of cardinality $r + 1$ inducing only red or blue edges. By the choice of r , U induces a red or a blue b -clique; select one and denote its vertex set by V_{i+1} .

Let V_q be the last selected set; set $V_0 = V(G) \setminus (\cup_{i=1}^q V_i)$. The stop condition implies

$$\frac{(r-1)|V_0|^2}{2r} \geq \binom{|V_0|}{2} - \frac{\varepsilon^2}{4r}n^2 > \frac{|V_0|^2}{2} - \frac{n}{2} - \frac{\varepsilon^2}{4r}n^2.$$

This, and $\varepsilon^2 n < 2r$, imply $|V_0| < \varepsilon n$. Every set V_1, \dots, V_q spans either a red or a blue b -clique, so the partition $V(G) = \cup_{i=0}^q V_i$ is as required. \square

We shall need also the following modification of Lemma 10.

Lemma 17 *Let $r \geq 1$, $0 < 2\varepsilon^{1/r} < d < 1 - 2\varepsilon^{1/r}$, and let (A, B) be an ε -uniform pair with $d(A, B) = d$. There are at most $\varepsilon 2^r |A|^r$ r -sets $R \subset A$ such that, there exists a partition $R = R_0 \cup R_1$ satisfying*

$$|(\cap_{u \in R_0} \Gamma(u)) \cap (\cap_{u \in R_1} (B \setminus \Gamma(u))) \cap B| \leq \varepsilon |B|.$$

Proof of Theorem 5 For $r = 2$ the assertion easily follows from the Scooping Lemma. To prove it for $r > 2$ we apply induction on r - assuming it holds for r , we shall prove it for $r + 1$.

It is sufficient to find $\xi = \xi(\varepsilon, r + 1) > 0$ and $L = L(\varepsilon, r + 1)$ such that, for every graph H of order $r + 1$, and every graph G of sufficiently large order n , if $k_H(G) < \xi n^{r+1}$, then there exists a partition $V(G) = \cup_{i=0}^q W_i$ such that:

(i) $q \leq L$;

(ii) $|W_0| < 6\varepsilon n$, $|W_1| = \dots = |W_q|$;

(iii) for every $i \in [q]$, $e(W_i) < \varepsilon \binom{|W_i|}{2}$ or $e(W_i) > (1 - \varepsilon) \binom{|W_i|}{2}$.

We shall outline our proof first. Choose b , δ , $\xi(\varepsilon, r + 1)$, and $L(\varepsilon, r + 1)$ appropriately. Select any graph H , let G be a graph of sufficiently large order n , and let $k_H(G) < \xi(\varepsilon, r + 1) n^{r+1}$. Applying SUL, we find a δ -uniform partition $V(G) = \cup_{i=0}^k V_i$. Note that, if $k_F(V_i)$ is proportional to n^r , and V_i is incident to a δ -uniform pair of medium density, then, by Lemma 17, there are substantially many induced copies of H in G . Therefore, for every V_i , either $k_F(V_i)$ is small, or V_i is incident only to very sparse or very dense δ -uniform pairs.

Let V' be the vertices in the clusters V_i with small $k_F(V_i)$; set $V'' = V \setminus (V' \cup V_0)$.

1 Partitioning of V'

By the induction hypothesis, we partition each V_i with small $k_F(V_i)$ into a bounded number of sets Y_{ij} that are either very sparse or very dense and a small exceptional set; the exceptional sets are collected in X' . Although the sets Y_{ij} are very sparse or very dense, they are not good for our purposes, for their cardinality may vary with i . To overcome this obstacle, we first select a sufficiently small integer s proportional to n . Then, by the Scooping Lemma, we partition each of the sets Y_{ij} into sparse or dense sets of cardinality exactly s and a small exceptional set; the exceptional sets are added to X' .

2 Partitioning of V''

We partition V'' into dense or sparse sets of size s and a small exceptional set X_0 . Suppose V_1, \dots, V_g are the clusters whose union is V'' . If $g \ll k$, we let $X_0 = V''$, and complete the partition, so suppose that g is proportional to k . The density of the clusters is unknown, so we assemble them into larger groups of b clusters. Recall that the pairs (V_i, V_j) ($1 \leq i < j \leq g$) are either very sparse, or very dense, or are not δ -uniform. Color correspondingly the edges of K_g in red, blue and green. Since, there are fewer than $\delta k^2 \leq \delta' g^2$ green edges, by Lemma 16, we assemble almost all clusters V_1, \dots, V_g into groups of exactly b clusters and collect the vertices of the few remaining clusters in a set X_0 . Observe that the pairs within the same group are all either very dense or all very sparse. Finally, we apply the Scooping Lemma to partition each of the groups into dense or sparse sets of cardinality s and an exceptional class; the exceptional classes are added to X_0 .

Let W_1, \dots, W_q be the sets of cardinality s obtained during the partitioning of V' and V'' . Setting $W_0 = V_0 \cup X_0 \cup X'$, the choice of δ implies $|W_0| < \varepsilon n$, so the partition $V(G) = \cup_{i=0}^q W_i$ satisfies (i)-(iii).

Let us give the details now. Let $\gamma(\varepsilon, b)$ and $n(\varepsilon, b)$ be as defined in Lemma 16, $M(\delta, l)$ as defined in SUL, and $\xi(\varepsilon^3, r)$, $L(\varepsilon^3, r)$ as defined in Theorem 3. Assume ε sufficiently small and set

$$b = \lceil \varepsilon^{-3} \rceil, \quad (37)$$

$$l = n(\varepsilon, b), \quad (38)$$

$$\delta = \min \left\{ \gamma(\varepsilon, b) \varepsilon^2, \frac{\xi(\varepsilon^3, r)}{2^r + 1}, \frac{\varepsilon^{3r}}{4^r} \right\} \quad (39)$$

$$L(\varepsilon, r+1) = \frac{8M(\delta, l) L(\varepsilon^3, r)}{\varepsilon}, \quad (40)$$

$$\xi = \xi(\varepsilon, r+1) = \frac{\delta^2}{(2M(\delta, l))^{r+1}}. \quad (41)$$

Select any graph H with $|H| = r+1$ and fix a vertex $v \in V(H)$. Set $F = H - v$ and let $F_1 = \Gamma(v)$, $F_0 = F \setminus F_1$.

Let G be a graph of sufficiently large order n , and let $k_H(G) < \xi n^{r+1}$. We apply SUL, and find a δ -uniform partition $V(G) = \cup_{i=0}^k V_i$ with $l \leq k \leq M(\delta, l)$. Set $t = |V_1|$ and observe that

$$\frac{n}{2k} \leq (1 - \delta) \frac{n}{k} < t \leq \frac{n}{k}. \quad (42)$$

Assume that there exist a cluster V_i with $k_F(V_i) > \xi(\varepsilon^3, r) t^r$, and a δ -uniform pair (V_i, V_j) with

$$1 - 2\delta^{1/r} > d(V_i, V_j) > 2\delta^{1/r}.$$

We apply Lemma 17 with $A = V_i$ and $B = V_j$, and find that there are at least

$$\xi(\varepsilon^3, r) t^r - \delta 2^r t^r \geq \delta t^r$$

induced subgraphs V_i isomorphic to F such that, if $X \subset G[V_i]$ and $\Phi : F \rightarrow X$ is an isomorphism, then

$$|(\cap_{u \in \Phi(F_0)} \Gamma(u)) \cap (\cap_{u \in \Phi(F_1)} (B \setminus \Gamma(u))) \cap B| > \delta |B|.$$

Hence, there are at least $\delta^2 t^{r+1}$ induced copies of H inducing a copy of F in V_i and having a vertex in V_j . Therefore, from (42) and (41), we find that

$$k_H(G) \geq \delta^2 t^{r+1} > \delta^2 \left(\frac{1-\delta}{k} \right)^{r+1} n^{r+1} > \frac{\delta^2 n^{r+1}}{(2M(\delta, l))^{r+1}} = \xi(\varepsilon, r+1) n^{r+1},$$

a contradiction. Therefore, if $k_F(V_i) > \xi(\varepsilon^3, r) t^r$, then every δ -uniform pair (V_i, V_j) satisfies

$$d(V_i, V_j) \leq 2\delta^{1/r} \quad \text{or} \quad d(V_i, V_j) \geq 1 - 2\delta^{1/r}$$

Let

$$I' = \{i : i \in [k], k_F(V_i) \leq \xi(\varepsilon^3, r) t^r\}, \quad I'' = [k] \setminus I'.$$

First we shall partition the set $V' = \cup_{i \in I'} V_i$. Set

$$s = \left\lfloor \frac{\varepsilon n}{4kL(\varepsilon^3, r)} \right\rfloor, \tag{43}$$

and observe that

$$\frac{n}{s} < \frac{8kL(\varepsilon^3, r)}{\varepsilon} < \frac{8M(\delta, l)L(\varepsilon^3, r)}{\varepsilon} = L(\varepsilon, r+1). \tag{44}$$

For every $i \in I'$, by the induction hypothesis, we find an equitable partition $V_i = \cup_{j=0}^{m_i} Y_{ij}$ with $|Y_{i0}| < m_i \leq L(\varepsilon^3, r)$ such that

$$e(Y_{ij}) < \varepsilon^3 \binom{|Y_{ij}|}{2} \quad \text{or} \quad e(Y_{ij}) > (1 - \varepsilon^3) \binom{|Y_{ij}|}{2}$$

for every $j \in [m_i]$. Also, for every $i \in I'$ and $j \in [m_i]$, (43) and (42) imply

$$s \leq \frac{\varepsilon n}{4kL(\varepsilon^3, r)} < \varepsilon \frac{t}{2L(\varepsilon^3, r)} \leq \varepsilon \frac{t}{2m_i} \leq \varepsilon \left\lfloor \frac{t}{m_i} \right\rfloor = \varepsilon |Y_{ij}|.$$

Hence, for every $i \in I'$ and $j \in [m_i]$, we apply the Scooping Lemma to the graph $G[Y_{ij}]$, and find a partition $Y_{ij} = \cup_{q=0}^{p_{ij}} W_{ijq}$ with $|W_{ij0}| < \lceil \varepsilon |Y_{ij}| \rceil$ such that

$$|W_{ijq}| = s, \quad \text{and} \quad e(W_{ijq}) < \varepsilon \binom{s}{2} \quad \text{or} \quad e(W_{ijq}) > (1 - \varepsilon) \binom{s}{2}$$

for every $q \in [p_{ij}]$. Setting

$$X' = (\cup_{i \in I} Y_{i0}) \cup (\cup_{i \in I} \cup_{j=1}^{m_i} W_{ij0}),$$

we obtain

$$\begin{aligned}
|X'| &= |(\cup_{i \in I} Y_{i0})| + |(\cup_{i \in I} \cup_{j=1}^{m_i} W_{ij0})| < \sum_{i \in I'} m_i + \sum_{i \in I'} \sum_{j=1}^{m_i} [\varepsilon |Y_{ij}|] \\
&< kL(\varepsilon^3, r) + 2\varepsilon \sum_{i \in I'} m_i \left\lfloor \frac{t}{m_i} \right\rfloor \leq kL(\varepsilon^3, r) + 2\varepsilon n < 3\varepsilon n.
\end{aligned} \tag{45}$$

Denote by h be the number of the sets W_{ijq} ($i \in I'$, $j \in [m_i]$, $q \in [p_{ij}]$), and renumber them sequentially from 1 to h . Thus we have a partition $V' = X' \cup (\cup_{i=1}^h W_i)$ with $|X'| \leq 2\varepsilon n$ such that

$$|W_i| = s, \quad \text{and} \quad e(W_i) < \varepsilon \binom{s}{2} \quad \text{or} \quad e(W_i) > (1 - \varepsilon) \binom{s}{2}$$

for every $i \in [h]$.

Next we shall partition the set $V'' = \cup_{i \in I''} V_i$. For convenience assume $I'' = [g]$; we may assume $g \geq \varepsilon k$, else, setting $W_0 = V_0 \cup X' \cup V''$, from (45), we have $W_0 < 4\varepsilon n$, and, in view of (44), the proof is completed.

Recall that the pairs (V_i, V_j) ($1 \leq i < j \leq g$) satisfy one of the following conditions:

- a) (V_i, V_j) is δ -uniform and $d(V_i, V_i) < 2\delta^{1/r}$;
- b) (V_i, V_j) is δ -uniform and $d(V_i, V_i) > 1 - 2\delta^{1/r}$;
- c) (V_i, V_j) is not δ -uniform.

Let K_g be the complete graph on the vertex set $[g]$; for every $1 \leq i < j \leq g$, color the edge (i, j) in red, blue or green correspondingly to a), b) and c). Observe that all pairs (V_i, V_j) ($1 \leq i < j \leq k$) that are not δ -uniform are fewer than

$$\delta k^2 < \frac{\delta}{\varepsilon^2} g^2 \leq \frac{\delta}{\varepsilon^2} g^2 \leq \gamma(\varepsilon, b) g^2,$$

so, the green edges are fewer than $\gamma(\varepsilon, b) g^2$. We apply Lemma 16, and find a partition $[g] = \cup_{i=0}^a X_i$ with $|X_0| < \varepsilon g$, and X_i is either a red or a blue b -clique for every $i \in [a]$. For every $j = 0, 1, \dots, a$, set $Y_j = \cup_{i \in X_j} V_i$; thus

$$|Y_0| < \varepsilon g t \leq \varepsilon n, \quad |Y_1| = \dots = |Y_a| = bt. \tag{46}$$

Fix some $c \in [a]$ and assume X_c a red b -clique. This is to say that all pairs (V_i, V_j) , $(i, j \in X_c, i < j)$, are δ -uniform and $d(V_i, V_i) < 2\delta^{1/r}$. Hence, from $|Y_c| = bt$, (37), and (39), we deduce

$$\begin{aligned}
e(Y_c) &= \sum_{i \in X_c} e(V_i) + \sum_{i, j \in X_c, i < j} e(V_i, V_j) \leq b \binom{t}{2} + \binom{b}{2} 2\delta^{1/r} t^2 \\
&< \frac{bt^2}{2} + \delta^{1/r} b^2 t^2 = \left(\frac{1}{2b} + \delta^{1/r} \right) |Y_c|^2 \leq \left(\frac{\varepsilon^3}{4} + \frac{\varepsilon^3}{4} \right) |Y_c|^2 = \varepsilon^3 \binom{|Y_c|}{2}.
\end{aligned}$$

On the other hand, (43) and (46) imply

$$s \leq \frac{\varepsilon n}{4kL(\varepsilon^3, r)} = \frac{\varepsilon}{4L(\varepsilon^3, r)} t < \varepsilon bt = \varepsilon |Y_c|.$$

Hence, we apply the Scooping Lemma to the graph $G[Y_c]$, and find a partition $Y_c = \cup_{i=0}^{f_c} Z_{ci}$ with $|Z_{c0}| < s$ such that

$$|Z_{ci}| = s, \quad \text{and} \quad e(Z_{ci}) < \varepsilon \binom{s}{2}$$

for every $i \in [f_c]$.

If X_c is a blue b -clique, proceeding in a similar way, we find a partition $Y_c = \cup_{i=0}^{f_c} Z_{ci}$ with $|Z_{c0}| < s$ such that

$$|Z_{ci}| = s, \quad \text{and} \quad e(Z_{ci}) > (1 - \varepsilon) \binom{s}{2}$$

for every $i \in [f_c]$. Set $X' = Y_0 \cup (\cup_{i=1}^a Z_{i0})$ and observe that (43) and (46) imply

$$\begin{aligned} |X'| &\leq |Y_0| + \sum_{i=1}^a |Z_{i0}| \leq \varepsilon n + sa < \varepsilon n + \frac{\varepsilon n}{4kL(\varepsilon^3, r)} \frac{k}{b} \\ &< \left(\varepsilon + \frac{\varepsilon^4}{4L(\varepsilon^3, r)} \right) n < 2\varepsilon n. \end{aligned} \quad (47)$$

Denote by h' the number of the sets Z_{ci} ($c \in [a]$, $i \in [f_c]$), and renumber them from 1 to h' . Set

$$\begin{aligned} q &= h + h', \\ W_0 &= V_0 \cup X_0 \cup X', \\ W_{h+i} &= Z_i, \quad i \in [h']. \end{aligned}$$

From (45) and (47) it follows

$$|W_0| = |V_0| + |X_0| + |X'| < M(\delta, l) + 2\varepsilon n + 3\varepsilon n < 6\varepsilon n.$$

Finally, (44) implies

$$q = h + h' \leq \frac{n}{s} < L(\varepsilon, r + 1),$$

completing the proof □

Applying the same argument as in the proof of Theorem 12, we enhance Theorem 5 as follows.

Theorem 18 *For all $\varepsilon > 0$, $r \geq 2$ and $k \geq 2$, there exist $\rho = \rho(\varepsilon, r, k) > 0$ and $K = K(\varepsilon, r, k)$ such that, for every graph G of sufficiently large order n , the following assertion holds.*

If $k_r(G) \leq \rho n^r$, then there exists an ε -uniform partition $V(G) = \cup_{i=0}^q V_i$ with $k \leq q \leq K$, and

$$e(V_i) < \varepsilon \binom{|V_i|}{2} \quad \text{or} \quad e(V_i) > (1 - \varepsilon) \binom{|V_i|}{2}$$

for every $i \in [q]$.

Acknowledgement. The author is grateful to Cecil Rousseau for his kind attention, and especially to Béla Bollobás for his valuable suggestions and remarks.

References

- [1] N. Alon, Bipartite subgraphs, *Combinatorica* **16** (1996), 301–311.
- [2] N. Alon, B. Bollobás, M. Krivelevich and B. Sudakov, Maximum cuts and judicious partitions in graphs without short cycles, *J. Comb. Theory Ser. B* **88** (2003), 329–346.
- [3] N. Alon, M. Krivelevich and B. Sudakov, MaxCut in H-free graphs, preprint.
- [4] B. Bollobás, *Extremal Graph Theory*, Academic Press Inc., London-New York, 1978, xx+488 pp.
- [5] B. Bollobás, *Modern graph theory*, Graduate Texts in Mathematics, **184**, Springer-Verlag, New York (1998), xiv+394 pp.
- [6] B. Bollobás and A. D. Scott, Exact bounds for judicious partitions of graphs, *Combinatorica* **19** (1999), 473–486.
- [7] B. Bollobás and A. D. Scott, Problems and results on judicious partitions, *Rand. Struct. Alg.* **21** (2002), 414–430.
- [8] P. Erdős, Some old and new problems in various branches of combinatorics, *Proceedings of the Tenth Southeastern Conference on Combinatorics, Graph Theory and Computing* (Florida Atlantic Univ., Boca Raton, Fla., 1979), *Congress. Numer.* **XXIII–XXIV**, Utilitas Math., Winnipeg, Man., 1979, pp. 19–37.
- [9] J. Komlós and M. Simonovits, Szemerédi’s regularity lemma and its applications in graph theory, *Combinatorics, Paul Erdős is Eighty*, Vol. **2** (Keszthely, 1993), pp. 295–352, Bolyai Soc. Math. Stud., 2, János Bolyai Math. Soc., Budapest, 1996.
- [10] Y. Kohayakawa and V. Rödl, Szemerédi’s regularity lemma and quasi-randomness, *Recent advances in algorithms and combinatorics*, CMS Books Math./Ouvrages Math. SMC, **11**, Springer, New York, 2003, pp. 289–351.
- [11] V. Nikiforov, On the edge distribution of a graph, *Comb. Prob. Comp.* **10** (2001), 543–555.